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Stability of the charged particle configuration of a nonlinear electromagnetic field

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Abstract. The stability of the configuration of a non-linear electromagnetic field corresponding to a charged point source is studied. The static classical solution of the field equations, describing charged particles has been modified by a small perturbation obeying linearised field equations. Assuming time dependence of the perturbation in the form $\exp(-i\omega t)$ it has been shown that ω^2 is non-negative and the classical static configuration of the field describing charged particles is stable. Detailed analysis of the linearised field equations has been performed for the Born–Infeld model of the non-linear electromagnetic field. There is only a continuous spectrum of ω^2 with certain distinct values of ω corresponding to the resonant states. The problem of the zero modes connected with translational invariance is also studied.

1. Introduction

Recently there has been a great deal of interest in the classical solutions of non-linear field equations (Jackiw 1977, t'Hooft 1974, Nielsen and Olesen 1973, Beliavin *et al* 1975). The relevance of this type of solution to elementary particle physics is still uncertain although there is some hope that they might describe localised particle states and provide some insight into the non-perturbative structure of quantum field theory (Jackiw 1977).

Classical solutions of the field equations, being solutions of the Euler–Lagrange equations of a variational problem, give stationary values of the action functional. If they are to describe stable particles the action functional should have a local minimum for these solutions, i.e. the second variation of the action should be non-negative. To examine stability one modifies the classical solution by a small perturbation with the exponential time dependence $\exp(-i\omega t)$. This perturbation fulfills linearised field equations with ω^2 as an eigenvalue. If the spectrum does not contain negative eigenvalues the classical solution is stable, i.e. small perturbations do not lead to solutions either growing or decaying in time.

In this paper we are going to investigate the stability of the configuration of a non-linear electromagnetic field corresponding to a charged point source. In general, any non-linear model of the electromagnetic field can be described by a Lagrange function of the type $\mathscr{L} = \mathscr{L}(S, P)$, where $\mathscr{L}$ is an, in principle arbitrary, function of two variables $S = (\frac{1}{2})(\boldsymbol{E}^2 - \boldsymbol{B}^2)$, $P = \boldsymbol{E} \cdot \boldsymbol{B}$. Due to a non-linear dependence of the induction $\boldsymbol{D}$ on the electric field $\boldsymbol{E}$ a model of this type can possess Coulomb-like configurations of the field with finite energy, in contrast with the linear Maxwell theory.

We are here dealing with one particular model of a non-linear electromagnetic field formulated a long time ago by Born and Infeld (1934a, b). The Lagrange function of this model has the following form

$$\mathscr{L}=b^2\left[1-\left(1-\frac{2S}{b^2}-\frac{P^2}{b^4}\right)^{1/2}\right] \tag{1.1}$$

where b is a parameter with the dimension of the field. Although this Lagrangian is different from the realistic effective Lagrangian in quantum electrodynamics (Heisenberg and Euler 1936, Schwinger 1951), it nevertheless describes a very interesting model of non-linear phenomena in the electromagnetic field. The main reason is that the Born–Infeld theory seems to play a prominent role among various possible models with a Lagrangian $\mathscr{L}(S, P)$.

First of all, as has been shown by Boillat (1970) and Plebański (1970), this model of a non-linear electromagnetic field is the only one in which double refraction of the plane electromagnetic waves does not occur. Another advantage of this theory is its considerable simplicity which, however, does not exclude various interesting features and, on the other hand, allows to find a Coulomb solution in a closed form. This is not the case, for instance, for the Heisenberg–Euler Lagrangian where due to its complexity $\boldsymbol{E}(\boldsymbol{D})$ cannot be found in a closed form. For these reasons we found it interesting to investigate the stability of the Coulomb solution in the Born–Infeld electrodynamics.

The paper is organised as follows. In § 2 we give a general description of the non-linear electromagnetic field. Section 3 deals with the problem of stability of the Coulomb solution and § 4 is devoted to the description of the zero modes connected with translational invariance. Section 5 contains discussion of the results.

2. General description of the non-linear electromagnetic field

As we have already mentioned in the introduction the non-linear electromagnetic field can be described by a Langrangian of the following general form:

$$\mathscr{L}=\mathscr{L}(S, P) \tag{2.1}$$

where

$$S=-\tfrac{1}{4}f_{\mu\nu}f^{\mu\nu}=\tfrac{1}{2}(\boldsymbol{E}^2-\boldsymbol{B}^2) \tag{2.2}$$
$$P=-\tfrac{1}{4}f_{\mu\nu}\check{f}^{\mu\nu}=\boldsymbol{E}\cdot\boldsymbol{B}$$

and $\check{f}_{\mu\nu}$ is the dual field $\frac{1}{2}\epsilon_{\mu\nu\lambda\rho}f^{\lambda\rho}$. Field equations have the form

$$f_{[\mu\nu,\lambda]}=0 \qquad h^{\nu\mu}{}_{,\mu}=j^{\nu} \tag{2.3}$$

where

$$h^{\mu\nu}(x)=-\frac{\partial\mathscr{L}}{\partial f_{\mu\nu}(x)}=\frac{\partial\mathscr{L}}{\partial S}f^{\mu\nu}(x)+\frac{\partial\mathscr{L}}{\partial P}\check{f}^{\mu\nu}(x). \tag{2.4}$$

Together with the fields $\boldsymbol{E}$, $\boldsymbol{B}$ one introduces the fields $\boldsymbol{D}$, $\boldsymbol{H}$ given by

$$D^i=-h^{0i} \qquad H^i=-\tfrac{1}{2}\epsilon^{ijk}h^{jk} \tag{2.5}$$

and therefore

$$D^i=\partial\mathscr{L}/\partial E^i \qquad H^i=\partial\mathscr{L}/\partial B^i. \tag{2.6}$$

In the three-dimensional notation, field equations without sources have the form

$$\nabla \cdot \boldsymbol{D} = 0 \qquad \dot{\boldsymbol{D}} = \nabla \times \boldsymbol{H}$$
$$\nabla \cdot \boldsymbol{B} = 0 \qquad \dot{\boldsymbol{B}} = -\nabla \times \boldsymbol{E}. \tag{2.7}$$

The first two equations follow from the variational principle whereas the last two are fulfilled identically due to potentials $f_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}$. For a general Lagrangian (2.1) the connection between $\boldsymbol{E}, \boldsymbol{B}$ and $\boldsymbol{D}, \boldsymbol{H}$ is non-linear and therefore the equations (2.7) form a system of coupled partial non-linear equations.

The Hamiltonian density is given by

$$\mathscr{H}(x) = \boldsymbol{E} \cdot \boldsymbol{D} - \mathscr{L} = \boldsymbol{E}\, \partial\mathscr{L}/\partial\boldsymbol{E} - \mathscr{L}. \tag{2.8}$$

If we choose $\boldsymbol{D}$ and $\boldsymbol{B}$ as canonical variables then we have the correspondence $p \to \boldsymbol{D}$, $q \to \boldsymbol{B}$ and $\boldsymbol{E}$ is the analogue of the velocity $\dot{q}$. In terms of canonical variables we have

$$E^i = \partial\mathscr{H}/\partial D^i, H^i = \partial\mathscr{H}/\partial B^i \tag{2.9}$$

and the dynamical field equations can be written as canonical equations:

$$\dot{\boldsymbol{D}} = \nabla \times \delta H/\delta \boldsymbol{B} \qquad \dot{\boldsymbol{B}} = -\nabla \times \delta H/\delta \boldsymbol{D} \qquad \left(H = \int \mathscr{H}\, \mathrm{d}^3x\right). \tag{2.10}$$

In any model of a non-linear electromagnetic field there exists a static solution describing the electric field of a point charge at rest. This is the spherically symmetric singular solution of the equation $\nabla \cdot \boldsymbol{D} = 0$:

$$\boldsymbol{D}_{\mathrm{cl}} = \frac{e}{4\pi} \frac{\boldsymbol{r}}{r^3} \tag{2.11}$$

and $\nabla \cdot \boldsymbol{D}_{\mathrm{cl}} = 0$ everywhere except at the origin:

$$\nabla \cdot D_{\mathrm{cl}} = e\delta^{(3)}(\boldsymbol{r}). \tag{2.12}$$

$\boldsymbol{D}_{\mathrm{cl}}$ describes the electric field of a point charge and is a solution of the field equations in a theory with a delta-like source.

For a Lagrangian (2.1) we have

$$\begin{aligned} \boldsymbol{D} &= \Lambda \boldsymbol{E} + \Omega \boldsymbol{B} \\ \boldsymbol{H} &= \Lambda \boldsymbol{B} - \Omega \boldsymbol{E} \end{aligned} \tag{2.13}$$

where

$$\Lambda = \partial\mathscr{L}/\partial S \qquad \Omega = \partial\mathscr{L}/\partial p. \tag{2.14}$$

For a purely electric field Λ is a function of $\boldsymbol{E}$ and $\boldsymbol{E}$ can be found by solving the first of the equations (2.13). The energy of such a field configuration is given by

$$\epsilon = \int [\boldsymbol{E}_{\mathrm{cl}}(\boldsymbol{D}_{\mathrm{cl}})\boldsymbol{D}_{\mathrm{cl}} - \mathscr{L}(\boldsymbol{D}_{\mathrm{cl}})]\, \mathrm{d}^3x. \tag{2.15}$$

For the linear theory (2.15) is divergent. However, one can choose $\mathscr{L}$ to make the energy finite. One such possibility is given by the Born–Infeld model of a non-linear electromagnetic field with the non-polynomial Lagrangian:

$$\mathscr{L}(x) = b^2\left[1 - \left(1 - \frac{2S}{b^2} - \frac{P^2}{b^4}\right)^{1/2}\right]. \tag{2.16}$$

b is a parameter with the dimension of the field and its inverse plays the role of a dimensional coupling constant. For $b \to \infty$ the coupling of the field to itself vanishes and $\mathscr{L}(x)$ tends to the Lagrangian of the Maxwell theory: $\mathscr{L}(x) = S$. The Hamiltonian density has the form:

$$\mathscr{H}(x) = b^2 \left[\left(1 + \frac{\boldsymbol{B}^2 + \boldsymbol{D}^2}{b^2} + \frac{(\boldsymbol{B} \times \boldsymbol{D})^2}{b^4} \right)^{1/2} - 1 \right] \tag{2.17}$$

and the energy is positive definite. The electric field corresponding to (2.11) is, in this theory, given by

$$\boldsymbol{E}_{\mathrm{cl}} = \frac{\boldsymbol{D}_{\mathrm{cl}}}{[1 + (D_{\mathrm{cl}}^2/b^2)]^{1/2}} = \frac{e}{4\pi} \frac{\boldsymbol{r}}{r^3} [1 + (l^4/r^4)]^{-1/2} \tag{2.18}$$

where

$$l = \left(\frac{|e|}{4\pi b} \right)^{1/2} \tag{2.19}$$

has the dimension of length. We see that there is a natural length scale in the theory.

According to (2.17) and (2.15) the energy of the field (2.18) is given by

$$\epsilon = b^2 \int \left[\left(1 + \frac{D_{\mathrm{cl}}^2}{b^2} \right)^{1/2} - 1 \right] \mathrm{d}^3x = \frac{e^2}{4\pi l} \int_0^\infty [(\lambda^4 + 1)^{1/2} - \lambda^2] \, \mathrm{d}\lambda \tag{2.20}$$

($\lambda = r/l$). This integral is convergent and the energy of the Coulomb field is finite.

3. Small oscillations around the classical solution

In this section we shall consider small oscillations around the classical solution given by (2.11) and (2.13). Due to the considerable simplicity of the Born–Infeld model of the electromagnetic field the calculations can be, to a large extent, performed analytically.

To proceed, let us assume that the classical solution has been perturbed by a small time-dependent field:

$$D_i = D_i^{\mathrm{cl}} + d_i \qquad H_i = h_i \tag{3.1}$$

and

$$E_i = E_i^{\mathrm{cl}} + e_i \qquad B_i = b_i. \tag{3.2}$$

In terms of the field tensor we have

$$f_{\mu\nu} = f_{\mu\nu}^{\mathrm{cl}} + \phi_{\mu\nu} \tag{3.3}$$

and we must expand $\boldsymbol{D}$ and $\boldsymbol{H}$ up to terms linear in $\boldsymbol{e}$ and $\boldsymbol{b}$. From (2.13) and (2.14) it now follows that

$$D_i = D_i^{\mathrm{cl}} + G_{ij} e_j \tag{3.4}$$

where

$$G_{ij} = \Lambda_{\mathrm{cl}} \left(\delta_{ij} + \frac{D_i^{\mathrm{cl}} D_j^{\mathrm{cl}}}{b^2} \right) \tag{3.5}$$

and

$$\Lambda_{\rm cl}=\frac{\partial\mathcal{L}}{\partial S}\bigg|_{\boldsymbol{B}=0}=\left(1-\frac{2S}{b^2}\right)^{-1/2}=\left(1+\frac{l^4}{r^4}\right)^{1/2} \tag{3.6}$$

where l is given by (2.19).

In the same way we get for the magnetic field $\boldsymbol{h}(=\boldsymbol{H})$.

$$h_i=\frac{1}{2}\frac{\Lambda_{\rm cl}}{b^2}E_i^{\rm cl}E_j^{\rm cl}\,\epsilon_{pqj}\,\phi_{pq}-\tfrac{1}{2}\Lambda_{\rm cl}\epsilon_{pqi}\phi_{pq} \tag{3.7}$$

where ϕ_{pq} is the spatial part of the field tensor $\phi_{\mu\nu}$. The equations $\boldsymbol{\nabla}\,.\,\boldsymbol{D}=0$ and $\dot{\boldsymbol{D}}=\boldsymbol{\nabla}\times\boldsymbol{H}$ now give

$$\partial_i(G_{ij}\phi_{0j})=0 \tag{3.8a}$$

$$G_{ij}\partial_0\phi_{0j}=\partial_m(G_{impq}\phi_{pq}) \tag{3.8b}$$

where

$$G_{impq}=\frac{1}{2}\frac{\Lambda_{\rm cl}}{b^2}\epsilon_{imn}\,e_{pqj}\,E_n^{\rm cl}E_j^{\rm cl}-\tfrac{1}{2}\Lambda_{\rm cl}\,(\delta_{ip}\delta_{mq}-\delta_{iq}\delta_{mp}). \tag{3.9}$$

Equation (3.8a) has to be valid every where except, possibly at the origin, for the same reasons that the equation $\boldsymbol{\nabla}\boldsymbol{D}^{\rm cl}=0$ is not fulfilled at the origin. Possible violation of the equation (3.8a) for $\boldsymbol{r}=0$ is connected with the vacuum polarisation which leads to the renormalisation of the charge e. This point will be discussed later.

Assuming now the time dependence in the form

$$\phi_{\mu\nu}(x)=\psi_{\mu\nu}(\boldsymbol{r})\,\mathrm{e}^{-\mathrm{i}\omega t} \tag{3.10}$$

we see that, due to the antisymmetry properties of G_{impq}, equation (3.8a) is a consequence of (3.8b). Equations (3.8) could also be obtained with an expansion of the Lagrangian up to terms quadratic in $\phi_{\mu\nu}$ or of the Hamiltonian up to terms quadratic in $\boldsymbol{b}$ and $\boldsymbol{d}$.

Owing to the spherical symmetry of the classical solution (2.11) and (2.18), equation (3.8) can be solved in the spherical coordinates. Angular part will be expressed in terms of the vector spherical functions

$$\boldsymbol{Y}_{JM}^{(e)}(\boldsymbol{n})=\mathrm{i}[J(J+1)]^{-1/2}r\boldsymbol{\nabla}Y_{JM}(\boldsymbol{n}) \tag{3.11a}$$

$$\boldsymbol{Y}_{JM}^{(m)}(\boldsymbol{n})=-\mathrm{i}[J(J+1)]^{-1/2}(\boldsymbol{r}\times\boldsymbol{\nabla})Y_{JM}(\boldsymbol{n}) \tag{3.11b}$$

$$\boldsymbol{Y}_{JM}^{(0)}(n)=-\mathrm{i}\boldsymbol{n}Y_{JM}(\boldsymbol{n}). \tag{3.11c}$$

3.1. *Magnetic photons*

We shall consider the first magnetic component of small oscillations whose angular dependence is given by $\boldsymbol{Y}_{JM}^{(m)}$. Oscillations of the magnetic type can be considered separately since their parity is different from the parity of $\boldsymbol{Y}_{JM}^{(e)}$ and $\boldsymbol{Y}_{JM}^{(0)}$. We choose the gauge $\psi_0=0$ and then put

$$\boldsymbol{\psi}(\boldsymbol{r})=\boldsymbol{Y}_{JM}^{(m)}(\boldsymbol{n})f^{(m)}(r) \tag{3.12}$$

where $\boldsymbol{\psi}$ is the space-dependent part of the vector potential.

It can easily be shown that equation (3.8a) is fulfilled identically and

$$G_{impq}\phi_{pq}^{(m)} = 2G_{impq}\partial_p\psi_q \,\mathrm{e}^{-\mathrm{i}\omega t}. \tag{3.13}$$

Using equations (3.9) and (3.13) we get

$$G_{impq}\,\partial_p\psi_q = \frac{1}{2}\frac{\Lambda_{\mathrm{cl}}}{b^2}\,\epsilon_{imn}\epsilon_{pqj}\,E_n^{\mathrm{cl}}E_j^{\mathrm{cl}}\,f\,\partial_p Y_q^{(m)}$$
$$-\tfrac{1}{2}\Lambda_{\mathrm{cl}}\left((\partial_i Y_m^{(m)} - \partial_m Y_i^{(m)})f^{(m)} + (x_i Y_m^{(m)} - x_m Y_i^{(m)})\frac{f^{(m)\prime}}{r}\right). \tag{3.14}$$

For the right-hand side of (3.8b) we have

$$\partial_m(G_{impq}\,\partial_p\psi_q)$$
$$= \frac{1}{2}\frac{\Lambda_{\mathrm{cl}}}{b^2}f^{(m)}\frac{E_{\mathrm{cl}}^2}{r^2}\,\epsilon_{imn}\,\epsilon_{pqj}\,x_n x_j\partial_m\partial_p Y_q^{(m)} + \frac{1}{2}(\Lambda_{\mathrm{cl}}f^{(m)})'\,\frac{1}{r}\,Y_i^{(m)}$$
$$+\tfrac{1}{2}\Lambda_{\mathrm{cl}}f^{(m)}\nabla^2 Y_i^{(m)} + \tfrac{1}{2}\left(\Lambda_{\mathrm{cl}}\frac{f^{(m)\prime}}{r}\right)' rY_i^{(m)} + \Lambda_{\mathrm{cl}}\frac{f^{(m)\prime}}{r}\,Y_i^{(m)} \tag{3.15}$$

where the prime denotes differentiation with respect to r and $Y_i^{(m)} = Y_{JM,i}^{(m)}$. It is straightforward to show that

$$\nabla^2 Y_{JM,i}^{(m)} = -\frac{1}{r^2}J(J+1)\,Y_{JMi}^{(m)} \tag{3.16}$$

and

$$\epsilon_{imn}\epsilon_{pqj}x_n x_j\partial_m\partial_p Y_{JM,q}^{(m)} = [J(J+1)-1]\,Y_{JM,i}^{(m)}. \tag{3.17}$$

For the electric field we have

$$\phi_{0j} = -\mathrm{i}\omega Y_j^{(m)}f^{(m)}(r)\,\mathrm{e}^{-\mathrm{i}\omega t}. \tag{3.18}$$

Substituting (3.15) and (3.18) into (3.8b) and using (3.16) and (3.17) we obtain the radial equation for $f^{(m)}$:

$$\frac{\mathrm{d}^2 f^{(m)}}{\mathrm{d}r^2} + \frac{2}{r}\left(1-\frac{E_{\mathrm{cl}}^2}{b^2}\right)\frac{\mathrm{d}f^{(m)}}{\mathrm{d}r} + \left(\omega^2 - \frac{J(J+1)}{r^2} + \frac{J(J+1)-2}{r^2}\frac{E_{\mathrm{cl}}^2}{b^2}\right)f^{(m)} = 0. \tag{3.19}$$

Furthermore we substitute

$$f^{(m)} = (l^4+r^4)^{-1/4}R^{(m)} \tag{3.20}$$

and introduce the dimensionless variable $x = r/l$. For $R^{(m)}$ we get

$$-\frac{\mathrm{d}^2 R^{(m)}}{\mathrm{d}x^2} + \frac{J(J+1)x^4(x^4+1)+5x^4+2}{x^2(x^4+1)^2}R^{(m)} = \epsilon^2 R^{(m)} \tag{3.21}$$

where $\epsilon = \omega l$ is the dimensionless energy parameter. (3.21) is the Schrödinger-type equation with the potential $V^{(m)}$:

$$V^{(m)}(x) = \frac{J(J+1)x^4(x^4+1)+5x^4+2}{x^2(x^4+1)^2}. \tag{3.22}$$

This J-dependent potential behaves like $2/x^2$ for $x \to 0$ and $J(J+1)/x^2$ for $x \to \infty$. We see that the repulsive part of the potential for $x \to 0$ is independent of J and therefore there is no centrifugal barrier in the usual quantum-mechanical sense. Furthermore, $V^{(m)}$ is always positive but in general it is not monotonic.

Zeros of the derivative of $V^{(m)}$ are given by roots of the following equation:

$$J(J+1)x^{12}+15x^8-[J(J+1)-5]x^4+2=0. \tag{3.23}$$

This cubic equation for x^4 has no real positive roots for $J=1, 2, 3$ but for $J\geqslant 4$ the potential $V^{(m)}$ has one minimum and one maximum at positive values of x. Therefore, for $J\geqslant 4$ there will be a 'potential barrier' in the Schrodinger equation (4.5). Locations and values of the extrema are given in table 1.

Table 1.

J	$x_{\min}$	$V^{(m)}_{\min}$	$x_{\max}$	$V^{(m)}_{\max}$
4	0·641	12·10	0·816	12·27
5	0·541	15·13	0·897	17·05
6	0·484	18·05	0·928	22·95
7	0·447	20·94	0·947	29·90
8	0·416	23·81	0·959	37·86
9	0·391	26·67	0·967	47·84
10	0·372	29·52	0·972	56·82
11	0·353	32·36	0·978	67·81
12	0·340	35·21	0·982	79·80
14	0·315	40·89	0·986	106·79

We see that the depth of the potential well increases with increasing J and therefore for values of J larger than some critical value J_{crit} we may expect that a 'resonance' or a metastable excited state would appear. For resonant values of ϵ, $\epsilon=\epsilon_{\text{res}}$, the amplitude of the function $R^{(m)}$ inside the potential well will be much larger than the amplitude of the function outside. Approximate values of ϵ_{res} can be found as bound-state energies in the potential equal to $V^{(m)}$ up to $x=x_{\max}$ and equal to constant $=V^{(m)}_{\max}$ for $x>x_{\max}$. Resonant values of ϵ will depend on J giving a trajectory $\epsilon_{\text{res}}(J)$.

Since $V^{(m)}$ is positive for all values of J (3.21) does not have negative eigenvalues and therefore the magnetic component of small oscillations around classical solution does not contribute to any instability of a charged particle.

A solution which is regular for $x=0$ behaves like x^2, and for $x\to\infty$ we find easily that

$$f^{(m)} \underset{x\to\infty}{\sim} \mathrm{e}^{\mathrm{i}\epsilon x}/x. \tag{3.24}$$

Therefore magnetic photons contribute to the electromagnetic field in the wave zone.

3.2. *Electric and longitudinal photons*

We cannot now consider $\boldsymbol{Y}^{(e)}$ and $\boldsymbol{Y}^{(0)}$ separately since, due to their having the same parity, these components of the electromagnetic field mix in the equations. It is more convenient to use from the beginning the field tensor, not the potentials, and therefore we assume:

$$\phi_{0j}=-\mathrm{i}\omega[Y^{(e)}_{JMj}f^{(e)}(r)+Y^{(0)}_{JMj}f^{(0)}(r)]\,\mathrm{e}^{-\mathrm{i}\omega t} \tag{3.25a}$$

$$\phi_{ij}=(x_i\,Y^{(e)}_{JMj}-x_jY^{(e)}_{JMi})h(r)\,\mathrm{e}^{-\mathrm{i}\omega t}. \tag{3.25b}$$

It can be shown that (3.25b) is the most general form of the magnetic field given by the electric and longitudinal parts of the vector potential. Since we have not introduced the potentials we must require now that the equations $\nabla \cdot \boldsymbol{B} = 0$, $\dot{\boldsymbol{B}} = -\nabla \times \boldsymbol{E}$ are fulfilled. It can easily be shown that $\nabla \cdot \boldsymbol{b} = 0$ identically and from the second equation we have

$$h = \frac{F^{(e)\prime}}{r^2} + [J(J+1)]^{1/2} f^{(0)}/r^2 \tag{3.26}$$

where

$$F^{(e)} = rf^{(e)}. \tag{3.27}$$

Let us now consider the first equation (3.8a). Although this is not an independent equation we consider it separately since, as we shall see, it follows from (3.8a) that the longitudinal part of the field gives rise to vacuum polarisation. We have

$$\partial_i(G_{ij}\phi_{0j}) = (\partial_i G_{ij})\phi_{0j} + G_{ij}\partial_i\phi_{0j} \tag{3.28}$$

and

$$\partial_i G_{ij} = e\frac{\Lambda_{\mathrm{cl}}}{b^2} D_j^{\mathrm{cl}}\,\delta^{(3)}(\boldsymbol{r}) + \Lambda'_{\mathrm{cl}}\frac{x_j}{r} + \frac{x_j}{r}(rD_{\mathrm{cl}})^2\left(\frac{1}{r^2}\frac{\Lambda_{\mathrm{cl}}}{b^2}\right)'. \tag{3.29}$$

The delta function term is responsible for the vacuum polarisation and renormalisation of the charge. Although the coefficient at the delta function is infinite for $r \to 0$ it gives a finite contribution when multiplied by $f^{(0)}$.

From (3.25a) we have

$$G_{ij}\partial_i\phi_{0j} = -\mathrm{i}\omega \mathrm{e}^{-\mathrm{i}\omega t}\Lambda_{\mathrm{cl}}\left[-\frac{\mathrm{i}}{r^2}[J(J+1)]^{1/2}F^{(e)} - \frac{2\mathrm{i}}{r}f^{(0)} - \mathrm{i}\left(1+\frac{D^2_{\mathrm{cl}}}{b^2}\right)f^{(0)\prime}\right]Y_{JM}. \tag{3.30}$$

The radial part of the constraint equation (3.8a) for $r \neq 0$ now becomes

$$[J(J+1)]^{1/2}F^{(e)} + \left[2r + r^2\frac{\mathrm{d}\ln\Lambda_{\mathrm{cl}}}{\mathrm{d}r} + \frac{(r^2D_{\mathrm{cl}})^2}{\Lambda_{\mathrm{cl}}}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{r^2}\frac{\Lambda_{\mathrm{cl}}}{b^2}\right)\right]f^{(0)} + r^2\left(1+\frac{D^2_{\mathrm{cl}}}{b^2}\right)f^{(0)\prime} = 0. \tag{3.31}$$

We have further that

$$G_{ij}\phi_{0j} = -\mathrm{i}\omega\mathrm{e}^{-\mathrm{i}\omega t}\left(\Lambda_{\mathrm{cl}}Y_i^{(e)}f^{(e)} + \Lambda_{\mathrm{cl}}Y_i^{(0)}f^{(0)} + \frac{\Lambda_{\mathrm{cl}}}{b^2}D^2_{\mathrm{cl}}Y_i^{(0)}f^{(0)}\right) \tag{3.32a}$$

$$G_{impq}\phi_{pq} = -\Lambda_{\mathrm{cl}}(x_iY_m^{(e)} - x_mY_i^{(e)})h\mathrm{e}^{-\mathrm{i}\omega t}. \tag{3.32b}$$

Substituting these expressions into (3.8b) and comparing on both sides coefficients at $Y^{(e)}_{JM,i}$ and $Y^{(0)}_{JM,i}$ we get two first-order equations for h and $f^{(e)}$. Together with (3.26) and (3.31) we get

$$\frac{x^4+1}{x^2}\frac{\mathrm{d}H}{\mathrm{d}x} + 2xH + \epsilon^2\frac{x^4+1}{x^4}F^{(e)} = 0 \tag{3.33a}$$

$$\epsilon^2\frac{x^4+1}{x^4}\frac{\mathrm{d}F^{(e)}}{\mathrm{d}x} - \left(\frac{x^4+1}{x^4}\epsilon^2 - J(J+1)\right)H = 0 \tag{3.33b}$$

$$F^{(0)} = [J(J+1)]^{-1/2}\left(-\frac{\mathrm{d}F^{(e)}}{\mathrm{d}x} + x^2H\right) \tag{3.33c}$$

$$[J(J+1)]^{1/2}F^{(e)}+2\frac{x^4-2}{x^3}F^{(0)}+\frac{x^4+1}{x^2}\frac{\mathrm{d}F^{(0)}}{\mathrm{d}x}=0 \tag{3.33d}$$

where $H=l^3h, F^{(0)}=lf^{(0)}$ and $F^{(e)}=rf^{(e)}$. The constraint equation (3.33d) follows from the first three equations.

We can calculate $F^{(e)}$ from (3.33a) and substitute it into (3.33b), thus obtaining a second-order equation for H. Having found the solution of this equation we can find the two remaining functions $F^{(e)}$ and $F^{(0)}$. The second-order equation for H has the form:

$$\frac{\mathrm{d}^2H}{\mathrm{d}x^2}+\frac{2}{x}\frac{2x^4+1}{x^4+1}\frac{\mathrm{d}H}{\mathrm{d}x}+\left(\epsilon^2+2x^2\frac{x^4+5}{(x^4+1)^2}-\frac{J(J+1)x^2}{x^4+1}\right)H=0. \tag{3.34}$$

Substituting

$$H=x^{-1}(x^4+1)^{-1/4}\chi \tag{3.35}$$

we get for χ

$$-\frac{\mathrm{d}^2\chi}{\mathrm{d}x^2}+x^2\frac{J(J+1)x^4+[J(J+1)-5]}{(x^4+1)^2}\chi=\epsilon^2\chi. \tag{3.36}$$

The potential $V^{(e)}$ in this equation goes to zero when $x\to 0$ and there is no centrifugal barrier at all. For $J=1$ $V^{(e)}$ has negative minimum and positive maximum whereas for $J\geqslant 2$ $V^{(e)}\geqslant 0$ and has one maximum. Locations and values of the extrema are given in table 2.

Table 2.

J	$x_{\min}$	$V^{(e)}_{\min}$	$x_{\max}$	$V^{(e)}_{\max}$
1	0·673	−0·808	1·643	0·455
2	—	—	1·266	2·065
3	—	—	1·127	4·903
4	—	—	1·072	8·837
5	—	—	1·046	13·806
6	—	—	1·032	19·789
7	—	—	1·024	26·779
8	—	—	1·018	34·772
9	—	—	1·014	43·768
10	—	—	1·012	53·765
11	—	—	1·010	64·762
12	—	—	1·008	76·760

Again the depth of the potential well increases with increasing J and we may expect resonances lying on the trajectory $\epsilon_{\mathrm{res}}(J)$. The only value of J for which a negative eigenvalue, $\epsilon^2<0$, could occur is $J=1$. But it can be shown that the first and only eigenvalue for $J=1$ is $\epsilon=0$. It corresponds to the zero mode connected with translation invariance of the theory. Since for $J>1$ we evidently do not have negative eigenvalues, we come to the conclusion that the charged particle in the Born–Infeld model is described by a stable configuration of the electromagnetic field.

Let us now consider the asymptotic behaviour of the electric and longitudinal components of the field. For $x \to \infty$ we get easily

$$H \sim \frac{\mathrm{e}^{\mathrm{i}\epsilon x}}{x^2} \qquad F^{(0)} \sim \frac{\mathrm{e}^{\mathrm{i}\epsilon x}}{x^2} \qquad F^{(e)} = xf^{(e)} \sim \mathrm{e}^{\mathrm{i}\epsilon x}. \tag{3.37}$$

From (3.25) we see that H and $f^{(e)}$ contribute to the electromagnetic field in the wave zone whereas the longitudinal part $F^{(0)}$ falls off rapidly and does not contribute to the energy flux across a large sphere in the asymptotic region of space.

For small x we may assume a power series expansion for $F^{(e)}$ and H. We find then that H contains only even powers of x and $F^{(e)}$ contains only odd powers of x:

$$H = 1 - \frac{\epsilon^2}{6}x^2 + \left[-\frac{1}{2} + \frac{\epsilon^2}{4}\left(\frac{\epsilon^2}{30} + \frac{J(J+1)}{5\epsilon^2}\right)\right]x^4 + \ldots \tag{3.38a}$$

$$F^{(e)} = \frac{1}{3}x^3 - \left[\frac{\epsilon^2}{30} + \frac{J(J+1)}{5\epsilon^2}\right]x^5 + \ldots \tag{3.38b}$$

and therefore

$$F^{(0)} = [J(J+1)]^{-1/2}\left(-\frac{\epsilon^2}{6}x^4 + \ldots\right). \tag{3.38c}$$

3.3. *Vacuum polarisation around the classical solution*

We shall now study vacuum polarisation effects which lead to the renormalisation of the bare charge e in (2.11). The magnitude of the charge can be found from the equation

$$\boldsymbol{\nabla} \cdot \boldsymbol{D} = e\delta^{(3)}(\boldsymbol{r}). \tag{3.39}$$

We have now

$$D_i = D_i^{\mathrm{cl}} + G_{ij}\phi_{0j}. \tag{3.40}$$

For $\boldsymbol{r} \neq 0$, $\partial_i(G_{ij}\phi_{0j}) = 0$ and from (3.29) we see that this equation is not true for $\boldsymbol{r} = 0$. We have therefore

$$\boldsymbol{\nabla} \cdot \boldsymbol{D} = e\delta^{(3)}(\boldsymbol{r}) + e\frac{\Lambda^{-}}{b^2}D_j^{\mathrm{cl}}\phi_{0j}\delta^{(3)}(\boldsymbol{r}). \tag{3.41}$$

Only the longitudinal part of ϕ_{0j} contributes since $x_jY^{(e)}_{JM,j} = 0$. According to (3.25a) the polarisation charge for a partial wave with given J is equal to

$$-|e|\omega\mathrm{e}^{-\mathrm{i}\omega t}\frac{1}{b}\frac{l^2}{r^2}\Lambda f^{(0)}_{JM}\bigg|_{r=0}. \tag{3.42}$$

But $\Lambda = (1 + l^4/r^4)^{1/2} \sim r^{-2}$ for $r \to 0$ and $f^{(0)}_{JM} \sim r^4$. Therefore, expression (3.42) is finite and different from zero. The exact magnitude of the vacuum polarisation correction to the bare charge e depends on the normalisation of $f^{(0)}_{JM}$.

Note that the vacuum polarisation is induced only by the longitudinal component of the field $\phi_{\mu\nu}$ whereas the electric and magnetic components, which are transverse with respect to $\hat{\boldsymbol{r}}$, do not contribute to this phenomenon. On the other hand, the longitudinal field does not contribute to the electromagnetic field in the wave zone and cannot be detected by a device located in the asymptotic region of space. We may say that the longitudinal field forms the cloud of virtual photons surrounding the charged particle and giving rise to the charge renormalisation.

4. Zero modes

In any field theory possessing certain symmetry properties small oscillations around the classical solution have an eigenvalue equal to zero. The simplest example of a zero mode is the derivative of the kink solution in $1+1$ dimensional ϕ^4 theory (Jackiw 1977). In the Born–Infeld model discussed in this paper we may expect zero modes to be connected with rotational and translational invariance. However, since the classical solution $\boldsymbol{D}^{\text{cl}}$ is spherically symmetric, space rotations do not lead to new solutions. Therefore there will be no zero modes connected with rotational invariance.

On the other hand, translational invariance leads to a new solution $\boldsymbol{D}^{\text{tr}} = (e/4\pi)(\boldsymbol{r}-\boldsymbol{r}_0)/|\boldsymbol{r}-\boldsymbol{r}_0|^3$ with the singularity shifted to $\boldsymbol{r}_0$ and we may expect zero modes connected with this invariance. Solutions corresponding to the translational zero mode are given by $\partial_j D_i^{\text{cl}}$ for three independent directions of translation $j = 1, 2, 3$. Let us now express $\boldsymbol{D}^{\text{cl}}$ by $\boldsymbol{E}^{\text{cl}}$:

$$D_i^{\text{cl}} = E_i^{\text{cl}}(1 - E_{\text{cl}}^2/b^2)^{-1/2}. \tag{4.1}$$

Then we can easily show that

$$\partial_j D_i^{\text{cl}} = G_{ik}\partial_j E_k^{\text{cl}} \tag{4.2}$$

where G_{ik} is given by (3.5). From (4.2) we see that $\partial_j E_i^{\text{cl}}$ is a small oscillation around the classical solution since it is connected with the corresponding electric induction field in a proper way.

The explicit form of the zero mode solution can be easily found:

$$\partial_j E_i^{\text{cl}} = \frac{bl^2}{r(r^4+l^4)^{1/2}}\left(\delta_{ji} - \frac{x_j x_i}{r^2}\frac{3r^4+l^4}{r^4+l^4}\right). \tag{4.3}$$

There are no zero modes connected with magnetic photons since $V^{(m)} > 0$. We assume

$$\partial_j E_i^{\text{cl}} = \sum_{J,M} [f_{JM_j}^{(e)}(r) Y_{JM,i}^{(e)} + f_{JM_j}^{(0)} T_{JM,i}^{(0)}]. \tag{4.4}$$

Then

$$f_{JM_j}^{(e)} = \int \partial_j \boldsymbol{E}^{\text{cl}} \boldsymbol{Y}_{JM}^{(e)*} \,\mathrm{d}\hat{\boldsymbol{r}} \tag{4.5a}$$

$$f_{JM_j}^{(0)} = \int \partial_i \boldsymbol{E}^{\text{cl}} \boldsymbol{Y}_{JM}^{(0)*} \,\mathrm{d}\hat{\boldsymbol{r}}. \tag{4.5b}$$

Finally

$$f_{JM_j}^{(e)} = \frac{bl^2}{r(r^4+l^4)^{1/2}} \int Y_{JM,j}^{(e)*} \,\mathrm{d}\hat{\boldsymbol{r}} \tag{4.6a}$$

$$f_{JM_j}^{(0)} = -\frac{2bl^2 r^3}{(r^4+l^4)^{3/2}} \int Y_{JM,j}^{(0)*} \,\mathrm{d}\hat{\boldsymbol{r}}. \tag{4.6b}$$

Since $Y_j^{(e)} = -[J(J+1)]^{-1/2} r p_j Y_{JM}$ and $Y_j^{(0)} = -\mathrm{i}\hat{x}_j Y_{JM}$ we see that only $J = 1$ gives a non-vanishing contribution. Using further that

$$p_j = \frac{1}{2\mathrm{i}}\left[x_j, \frac{\boldsymbol{L}^2}{r^3}\right] \tag{4.7}$$

where $\boldsymbol{L}$ is the angular momentum operator, we find

$$f^{(e)}_{JM_j} = \mathrm{i}\frac{bl^2}{r(r^4+l^4)^{1/2}}[J(J+1)]^{1/2}\int \frac{x_j}{r} Y^*_{JM}\, \mathrm{d}\hat{r} \tag{4.8a}$$

$$f^{(0)}_{JM_j} = \mathrm{i}\frac{2bl^2r^3}{(r^4+l^4)^{3/2}}\int \frac{x_j}{r} Y^*_{JM}\, \mathrm{d}\hat{r}. \tag{4.8b}$$

Introducing now the dimensionless variable $x = r/l$ and $F^{(e)}_j = xf^{(e)}_j$, $F^{(0)}_j = lf^{(0)}_j$ we find

$$F^{(e)\prime}_{JM_j} = -[J(J+1)]^{1/2}F^{(0)}_{JM_j}. \tag{4.9}$$

Let us now consider the zero modes specifying equations (3.33) for the case $\epsilon = 0$. From (3.33b) we see that $H = 0$ (i.e. no magnetic field with the zero mode) and from (3.33c) we obtain $F^{(e)\prime} = -[J(J+1)]^{1/2}F^{(0)}$, in agreement with (4.9). Eliminating $F^{(e)}$ from (3.33d) and (3.33c) with $H = 0$ we find

$$\frac{\mathrm{d}^2F^{(0)}}{\mathrm{d}x^2} + 2\frac{2x^4-3}{x(x^4+1)}\frac{\mathrm{d}F^{(0)}}{\mathrm{d}x} - \frac{[J(J+1)-2]x^4-12}{x^2(x^4+1)}F^{(0)} = 0. \tag{4.10}$$

Introducing the variable

$$x^4 = -z \tag{4.11}$$

and then substituting

$$F^{(0)} = z^{3/4}(1-z)^{-3/2}\phi \tag{4.12}$$

we reduce (4.10) to the hypergeometric equation:

$$z(z-1)\frac{\mathrm{d}^2\phi}{\mathrm{d}z^2} + [\gamma - (\alpha+\beta+1)z]\frac{\mathrm{d}\phi}{\mathrm{d}z} - \alpha\beta\phi = 0 \tag{4.13}$$

with

$$\alpha = -\tfrac{1}{4}(J+2) \qquad \beta = \tfrac{1}{4}(J-1) \qquad \gamma = \tfrac{3}{4}. \tag{4.14}$$

Two independent solutions of this equation give, for $F^{(0)}$,

$$F^{(0)}_{JM} = \frac{x^3}{(x^4+1)^{3/2}}F\left(-\frac{J+2}{4}, \frac{J-1}{4}, \frac{3}{4}; -x^4\right) \tag{4.15a}$$

$$'F^{(0)}_{JM} = \frac{x^3}{(x^4+1)^{5/4}}F\left(-\frac{J+1}{4}, \frac{J}{4}, \frac{5}{4}; -x^4\right). \tag{4.15b}$$

Only solution (4.15a) with $J = 1$ has the proper behaviour at infinity and therefore we have, for the zero mode solution,

$$F^{(0)}_{1M} = N_{1M}\frac{x^3}{(x^4+1)^{3/2}}. \tag{4.16}$$

This expression is consistent with the zero mode solution (4.8b) obtained from $\partial_j E^{\mathrm{cl}}_i$. There is no dependence on the direction of the translation in (4.16) since the radial part of the zero mode is the same for $j = 1, 2, 3$.

5. Conclusions

We have investigated the problem of the stability of the charged particle in non-linear electrodynamics. Our main conclusion is that the Coulomb solution in the Born–Infeld model of a non-linear electromagnetic field is stable, i.e. small pertubations of this solution do not lead to the field configurations either growing or decreasing in time.

We also found some other interesting properties of the spectrum of small oscillations around the classical solution describing a charged particle. It turned out that the spectrum is continuous and corresponds to the Schrödinger equation with the potential possessing a barrier-like structure. Since the depth of the potential well increases with increasing total spin quantum number J we may expect that a sort of resonant state would appear with the energies ϵ_{res} depending on J.

One of the characteristic features of the spectrum is that the higher is total spin J the larger is the probability of formation of a quasi-bound state since the potential well becomes deeper with increasing J. This is not what we are accustomed to in ordinary quantum mechanics. Usually the higher the angular momentum l is (and therefore the centrifugal force) the larger the probability is that the particle could be found far from the origin (the wavefunction goes to zero for $r \to 0$ more rapidly for higher l).

However, in our case both the magnetic potential $V^{(m)}$ (3.22) and the electric potential $V^{(e)}$ (3.36) do not exhibit the usual behaviour of the effective potential in the radial Schrödinger equation. The repulsive barrier in $V^{(m)}$ for $x \to 0$ is of the form $2/x^2$ instead of the standard $J(J+1)/x^2$ behaviour. For $x \to 0$ the wavefunction goes to zero like x^2 independent of J. Consequently there is no tendency to pull the particle away with increasing J. For the electric potential $V^{(e)}$ there is no repulsive barrier at all and for $x \to 0$, $V^{(e)} \to 0$ like $J(J+1)x^2$. These considerations show that the usual intuitions taken from quantum mechanics cannot be applied and do not lead to proper qualitative results.

In the quantised version of the theory the resonances mentioned above would correspond to excited states of the charged particle; these states would be unstable against decay into a massless vector particle (the photon) and a charged particle in the ground state. For appropriately high J more than one resonance would appear and then we may have a 'cascade':

$$c^{**} \to c^* + \gamma \to c + 2\gamma$$

where c symbolises a charged particle. In the scattering sector there should appear resonances in the cross sections for $\epsilon = \epsilon_{\text{res}}$.

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